IB Mathematics HL
Internal Assessment

Solving the Königsberg Bridge Problem

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IB Student No: -
May, 2017

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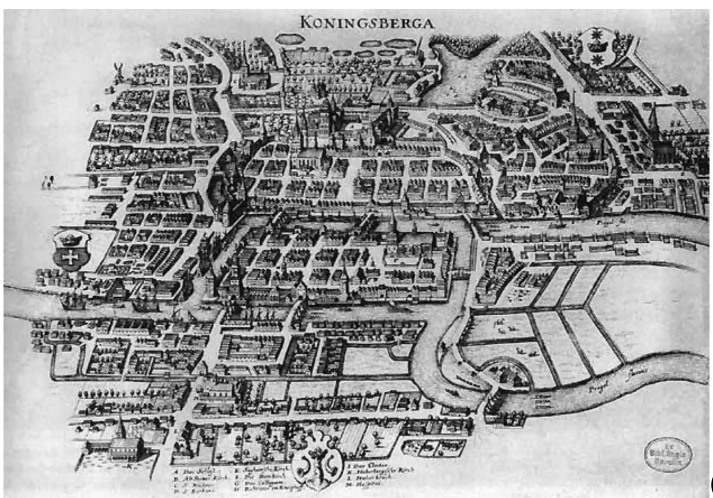
Introduction

I decided to explore the Königsberg Bridge Problem for my Internal Assessment. The problem first came to my attention in a video game I own where the player has to solve various logic problems in order to continue. One of these problems had to do with finding a path through an area that traversed every road once and only once. I did not realize it at the time but it was similar to the Königsberg Bridge Problem in that the requirements for the solution were the same and the setup of the roads was comparable to the setup of the bridges. After doing research on the topic, I learned that there was a mathematical way to solve the problem that stemmed from the branch of mathematics known as graph theory. By transforming the setup of the bridges into a graph, I could easily find whether or not there actually was a path and the exact path that could be taken to fulfill the requirements of the problem. This was the method used originally by Euler to find the solution to the Königsberg Bridge Problem back in the early 1700s when the problem was first presented.

Background

The Pregolya River runs through the city of Kaliningrad, Russia. In the past, the city was known as Königsberg and was a part of Germany. The river then split around an island in the middle of the city then briefly came back together before finally splitting into two separate rivers. In order to travel between landmasses, the people of Königsberg built seven bridges crossing the river.



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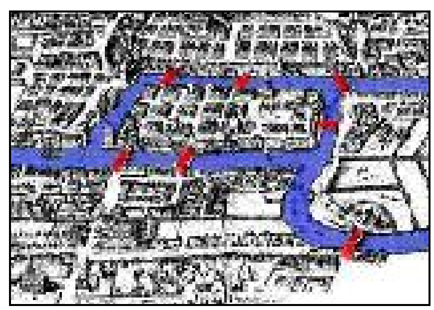
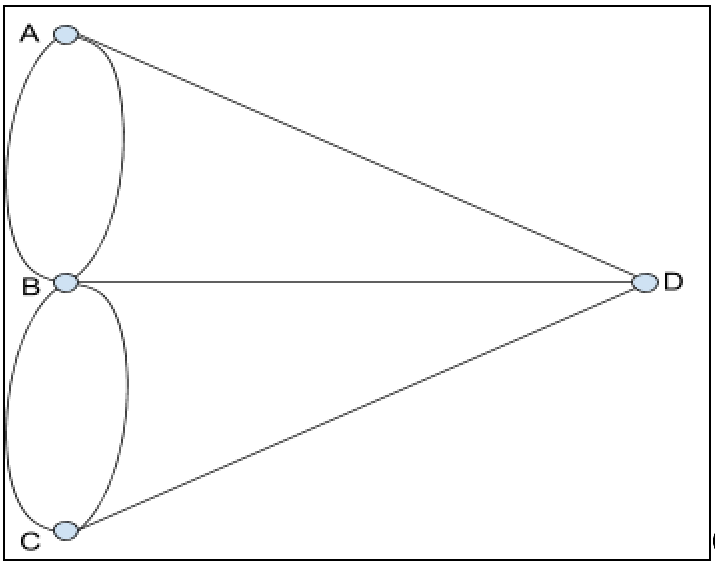

(2)

As a way of entertaining themselves, they devised a game where they tried to cross every one of the bridges once and only once, and some tried to both start and end at the same place as an extra challenge. However, none could accomplish either task no matter how much they tried. Eventually, the problem was posed to mathematician Leonhard Euler who tried and failed to solve it and who then declared that the problem actually had no solution at all.

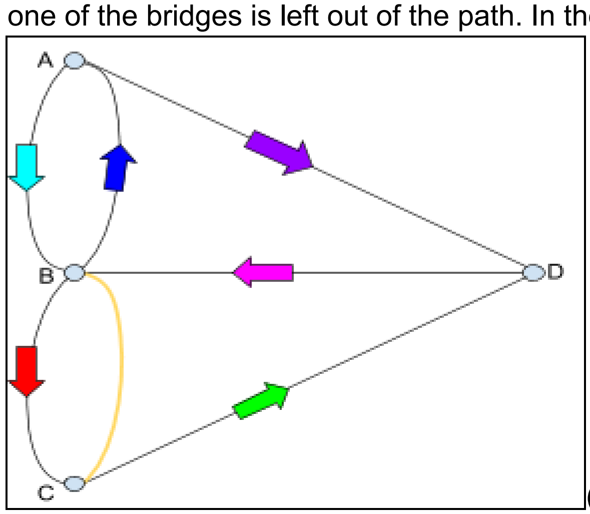
Euler's solution to the original bridge problem

The criteria for the original problem was to find a path across all seven bridges without crossing any bridge twice. Euler realized that trying to find a path by drawing the layout of the bridges and connecting them various ways would take a lot of time and would not necessarily result in a path that fulfilled the criteria. Instead, he made the problem into a graph problem. He made each bridge an edge, and each landmass became a node or vertex labeled by an uppercase letter (A through D). In the context of graphs, which are sets of vertices and edges, edges are lines that connect two nodes, while vertices or nodes are defined points of a graph, and the number of edges that connect to a vertex is that vertex's degree. The graph he made to represent the Königsberg bridges was similar to graph 1 pictured below.



(1)

In this graph, each of the four nodes have an odd degree which means they have an odd number of edges connected to them. Vertices A, C, and D all have a degree of three while only vertex B has a degree of five. If you start the path at one of the vertices with odd degree, you can visit each node but one of the bridges is left out of the path. In the example labeled graph 2,



(2)

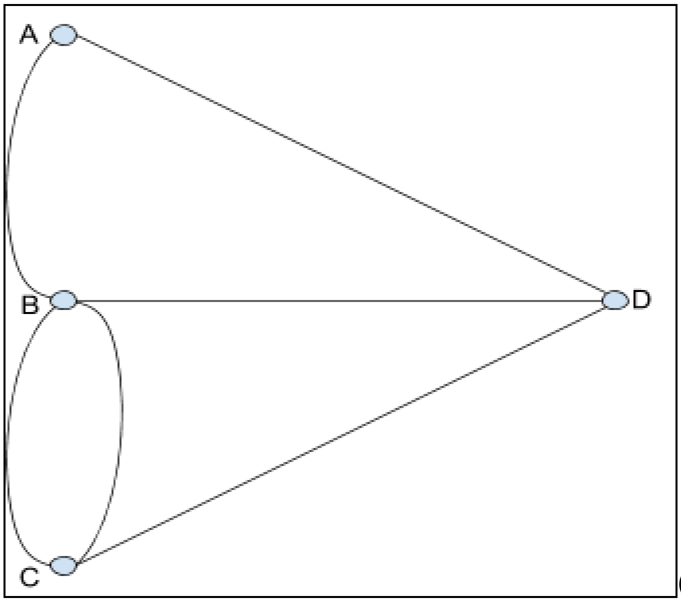
the path starts at node A and then continues to node B before returning to A, then on to D, then B, then C, before finally going back to D. However, with the path detailed, one edge is left out between nodes B and C (see highlighted edge) and so the criteria for the problem is not meet because each bridge is not crossed. The path can be started at a different node or travel in a different order between the nodes, but there will always be at least one edge not left out. This is because if an odd vertex is the starting point, you have to leave, then come back, then leave again for another node. If that other node is also an odd degree, then you have to leave after coming in and then come back again and there is no other edge to allow you to leave again for the other vertices. Thus if there are odd vertices, then the path must start and end at an odd vertex. However, there can only be one start and one end point so there can only be at most two vertices of odd degree in a graph. This graph has four vertices of odd degree so it is not possible to have a Eulerian path, or a path where each edge is used once and only once. Using this reasoning, Euler said that there is absolutely no path over the Königsberg bridges.

Application

Eventually, this problem and the solution or lack of solution to it became the start of graph theory along with the conclusions that came from the problem. Graph theory is a branch of mathematics that studies networks of points connected by lines. The Königsberg bridge problem was one of the first problems in this subject and led to the first theorems. The theorems in the terminology of modern graph theory state that “If there is a path along edges of a multigraph that traverses each edge once and only once, then there exist at most two vertices of odd degree” (Carlson). In this context, a multigraph is when any two vertices are connected by more than one edge, for example the graph created to represent the Königsberg bridges. A further extension of this theorem states that “if the path begins and ends at the same vertex, then no vertices will have odd degree” (Carlson). Together these statements create a foundation for modern graph theory. In addition, Euler’s work with the Königsberg bridge problem also led to the beginnings of the branch of mathematics known as topology or more specifically the topology of networks. The branch of topology studies properties of shapes that do not change when shapes are transformed while the topology of networks focuses on properties of networks that do not change like the number of edges and vertices and the degree of the vertices. Without Euler’s work on the bridges of Königsberg, graph theory and topology would not be what they are today as there would not be a foundation for the subjects.

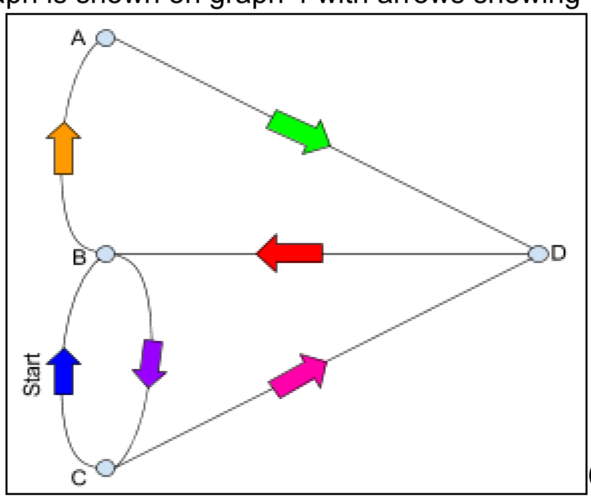
Alternate Versions of the Königsberg Bridge Problem

After researching and thinking about the solution and conclusion for the original Königsberg bridge problem, it came to my attention that the graph can be slightly changed so it is possible to solve. One of the changes would be to take away one bridge. Then there would be six edges between four nodes. One of the possible layouts if one edge was removed would look like the graph featured below labeled graph 3.



(3)

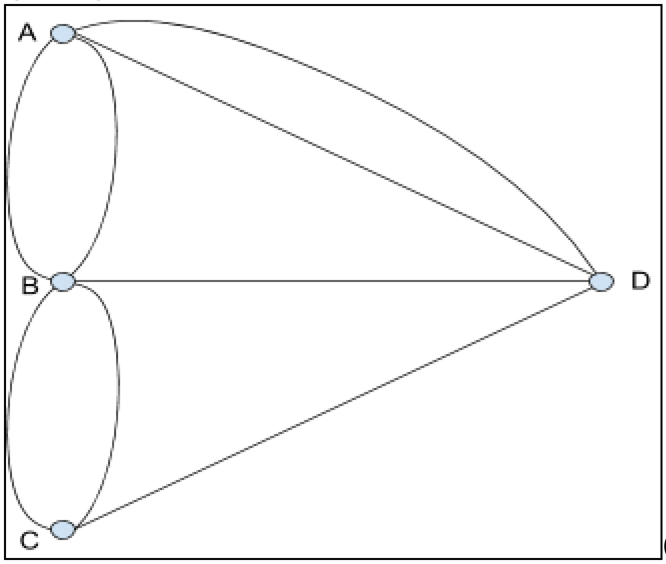
Here, one of the edges between nodes A and B has been removed. As a result, nodes A and B have an even degree of two and four respectively while nodes C and D each have an odd degree of three. Therefore there are only two vertices with an odd degree so a Eulerian path is possible for this network. However, for the path to cross each and every edge once and only once, the path must start at a node of odd degree and end at a node of odd degree. One of the possible paths for this graph is shown on graph 4 with arrows showing the direction of the path.



(4)

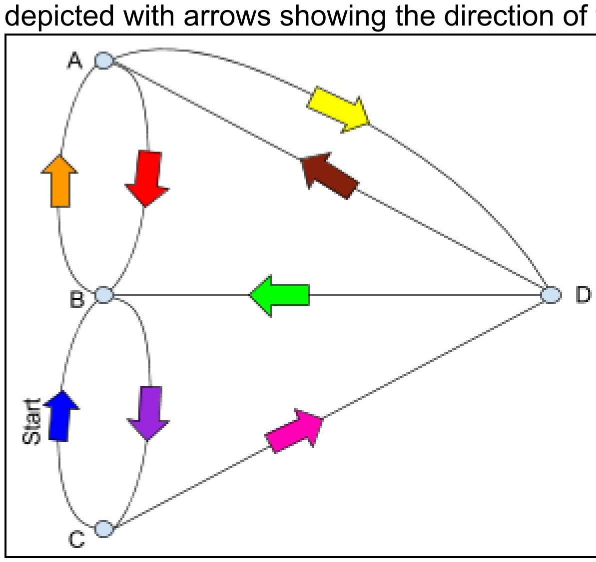
This specific path starts at node C then continues to node B, then A, then D, back to B, then C, before ending at node D. While this layout has proven to have a Eulerian path, any layout with one bridge removed would also have a Eulerian path as there would always be two nodes with an even degree and two nodes with an odd degree because of the layout of the original graph. In fact, if the outer edge between nodes A and B was removed, the resulting graph would be the same as the one featured. Also, if either one of the edges between nodes B and C was removed the resulting graph would simply be a reflection of the graph pictured so the path shown would still be possible so long as it was flipped vertically.

There are also other ways than removing one edge to alter the original graph so that there is a solution. One of them is to add an edge between nodes to the graph. This would result in a graph with eight edges and four nodes like the possible example in graph 5.



(5)

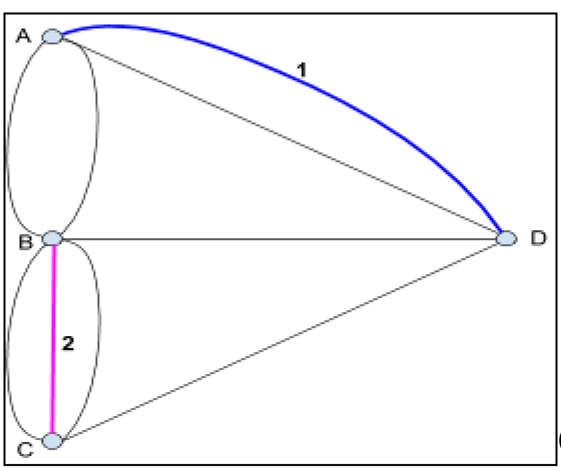
In this example, one edge has been added between nodes A and D. this causes both nodes A and D to have a degree of four while nodes C and B remain unaltered with a degree of three and five respectively. Thus this graph has only two nodes of odd degree with two nodes of even degree so according to the theorems of graph theory, there is a Eulerian path possible for this graph. However, like with the graph shown previously, the path must start at a node of odd degree and end at a node of odd degree. In the graph shown below and labeled graph 6, a possible Eulerian path is depicted with arrows showing the direction of the path.



(6)

The path begins at node C before going to B then back to C, then to node D, then A, before node B, then back to A, then D, before ending at node B. This is only one of the many possible paths for this graph and this is only one of the many possible graphs with one edge added to the original layout. The extra edge could have been added between any of the two nodes and still there would have been a Eulerian path because there would be two nodes with even degree and two with odd degree. If the eighth edge was instead added between nodes C and D, the resulting graph would be the reflection of the graph pictured here, and the path featured could also then be reflected to fit the resulting graph.

While adding or removing one edge results in a Eulerian path and satisfies the requirement of a path that crosses each bridge once and only once, it does not satisfy the additional requirement of a path that starts and ends at the same point. A path that crosses each edge once and only once and that starts and ends at the same point is called a Eulerian circuit or Eulerian cycle. For this to occur, no vertex of the graph can have an odd degree because if there is an odd degree, then the path has to either start or end there, but it cannot do both. If the path starts at the node with odd degree, the path leads out, comes back, then goes out again,but there is not another edge so it can come back and end there as well. If the path ends at a vertex of odd degree, it is a similar situation with pairs of edges being used to come in then go out and the remaining edge being used to come in at the end, but there is no other edge that could have been used to leave at the very beginning of the path which means the path started at another point. With the graph made from the Königsberg bridges, removing or adding only one edge is not enough to make all four vertices have an even degree. For that to occur, at least two bridges have to be added and at least two bridges have to be removed. For the added edges, the first edge can be added between any two nodes, but the second edge has to be added between the two nodes not connected by the first added edge so that all nodes than have an even degree. For instance in graph 7, the first edge can be added between nodes A and D, but nodes B and C still have odd degrees so the second edge has to be added between them. For the removed edges, any edge can be the first edge removed, but in order to end with all nodes having an even degree, the second edge removed has to be one of the edges between the nodes not connected to the first edge removed. For example in graph 8, the edge between vertices A and B is removed first, but nodes C and D still have odd degrees so the second edge removed has to be the one connecting those two nodes. If all nodes on the resulting graphs have an even degree, then the graph is a Eulerian circuit and a path can be started at any vertex and end at that same vertex thus fulfilling the extra requirement of the Königsberg bridge problem.



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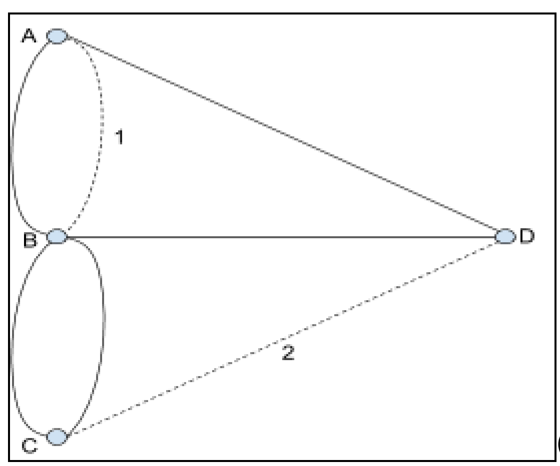
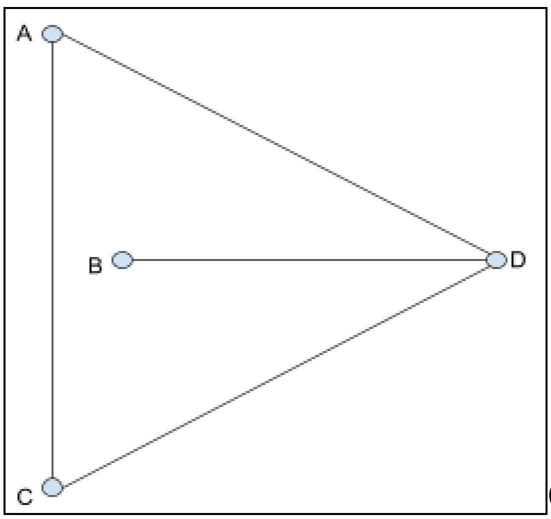


(8)

Conclusion

With the original layout of the seven bridges of Königsberg, it is impossible to find a path that crosses each and every bridge once as both the people of Königsberg discovered by trial and error and as Euler discovered using proofs based in the branch of mathematics known as graph theory. However, be adding or removing one or more bridges a path can be found and can depending on the number and choice of bridge result in a circuit being possible. Over time, the city of Königsberg became the city of Kaliningrad and the seven bridges crossing the Pregolya were altered so that they resembled those in graph 9. One of the bridges between nodes C and B was destroyed along with one of the bridges between nodes A and B. Also, the remaining bridges between C and B and between A and B were connected by a stairway above the island landmass that corresponds to node B. In this layout, there is still no way to start at one node and travel across all bridges once and end at the same node, or in other words no Eulerian cycle. However, there is now a Eulerian path possible if the path is started at node D, goes through nodes C and then A before returning to pass through D, and finally ends at node B as shown in the graph labeled 10.



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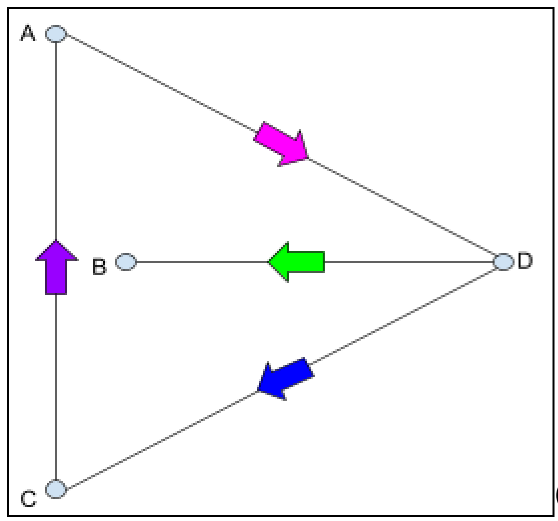

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Reflection

I enjoyed investigating the Königsberg bridge problem as it gave me the chance to learn more about a branch of mathematics I had previously had very little exposure to and about a simple solution to a problem I had encountered before. Learning about graph theory was interesting because I did not realize it was actually a viable branch of mathematics before this, and a very extensive and involved branch at that. I also found it interesting that a complicated logic problem like the Königsberg bridge problem could be solved so easily and quickly as whenever I came across a similar problem in the past, it took me a good amount of time to figure it out and sometimes I could not even find a solution at all. Additionally, I enjoyed finding out about the origins of the problem as I had not known anything about where the problem or the solution came from previously. Overall, by completing this investigation, I was able to not only gain a better understanding of certain mathematical topics, like graph theory and associated problems, but also expand my knowledge of mathematics and the history associated with it.

Image References

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